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ABSTRACT

This guide outlines an Option X course on problem solving techniques. The 16 learner objectives identified in the Mathematics Program Guide stress applications orientation to concepts about numbers and operations and common plane and solid figures. Techniques covered include working with diagrams, organizing information, using patterns, simplifying problems, and looking back. The document states that an activity designed for problem solving should not have a solution that is immediately apparent, and the guide provides a chapter reviewing each of the techniques listed above. Each chapter introduces the technique, states learner objectives, outlines an instructional strategy, gives sample activities, and concludes with a bibliography of suggested resources. Two appendices provide a list of problems to solve and list additional resources. (MP)

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OPTION X, LEVEL B

Problem Solving

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SE 034 610

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FOREWORD

In 1978 the Mathematics Program Guide, K-12 was developed and disseminated to all public schools in Hawaii "to provide direction for teachers and administrators in the development of school-level mathematics." One of the major outcomes of this effort was a substantial strengthening of the quantity and quality of the courses offered as part of the secondary mathematics program. Existing courses in grades 9-12 were restructured and several new courses were created. A course on problem solving, a critical goal of the mathematics program, was created so that students could receive instruction in problem solving in a course that was primarily applications oriented.

This document is a guide for the course on problem solving. The problems created by the teachers to be used as examples for the different techniques for problem solving should be applications to "real-life" situations such as business, consumerism, industry, and the trades. Students will have numerous opportunities to develop an in-depth knowledge of mathematical concepts as well as in applying the skills learned in areas such as those noted above.

The intent of this resource guide is to provide teachers with guidelines and materials in order to structure a course that would teach students mathematical content while learning problem solving techniques.

Charles G. Clark
Superintendent of Education

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UNIT I

Introduction

Course Overview

A primary focus of this course is on problem-solving techniques. These are sometimes referred to as "heuristics". Furthermore, the course will concentrate on having students solve problems that will enable them to attain the learner objectives of Option X, Level B courses. The learner objectives of Option X, Level B courses as listed on pages 124-125 of the Mathematics Program Guide are:

Numbers and Operations: Applied in Practical Situations

1. Solves ratio, proportion, and percent problems.
2. Adds, subtracts, multiplies, and divides integers.
3. Applies equation solving techniques to verbal problems.
4. Understands and uses the relationship among common fractions, decimals, and percents.
5. Applies properties of whole numbers.
6. Estimates the square roots of non-negative numbers.

Common Plane and Solid Figures: Applied in Practical Situations

7. Uses properties of similarity and congruence in constructions and applications.
8. Applies concepts of length, perimeter, area, volume, and angle measurement.
9. Applies trigonometric ratios in problem solving.
10. Applies the Pythagorean relation to problems involving right triangles.

Measurement: Applied in Practical Situations

11. Makes measurements using both customary and metric units.

12. Changes units within the metric system.

13. Computes perimeters, areas, and volumes using appropriate formulas.

Probability and Statistics: Applied in Practical Situations

14. Prepares and interprets simple statistically oriented graphs.

15. Collects and organizes numerical data.

16. Makes decisions after interpreting data.

Nature and Stages of Problems Solving

This course describes a problem as a situation that is new and unique for the person called upon to solve it. Hence what is a problem for one student may not be a problem for another. For example, finding a rule for the pattern: $1 + 2 + 3 \ldots + n$ might be a matter of recall for an algebra student but a problem for a student beginning this course.

Although knowledge of problem solving techniques does not guarantee the solution of problems, it does provide students with ideas to try. Infrequently we find students who when faced with a problem say, "I don't know what to do." Students will learn that there is a process that can be used to solve problems which has four stages: 1) understanding the problem, 2) devising a plan to solve the problem, 3) carrying out the plan, and 4) looking back or reviewing the problem and solution. There are various techniques of solving problems that could be used to help students proceed through each state of the problem solving process; however, only a few have been proposed, researched, and identified for investigation in this course due to time constraints which would not allow for the study of all techniques. The techniques covered in this course include: working with diagrams, organizing information,

using patterns, simplifying problems, and looking back.

Psychology of Problem Solving

Since not being able to obtain an immediate response to a problem situation is frustrating to some students, they should be made aware of the nature of problems. Namely, for a problem to exist the problem solution must not be immediately apparent. Furthermore, an activity would not provide the needed practice for problem solving if students had an immediate response. Thus, although students see the mathematics teacher solve problems "without difficulty," they should be made aware of the teacher's efforts at solving problems. In fact, it is part of the teacher's preparation for teaching.

Resources

Brownell, William. "Problem Solving." The Psychology of Learning. Part II of the Forty-first Yearbook of the National Society for the Study of Education. Chicago: The National Society for the Study of Education, 1942. pp. 415 - 443.

Cohen, L.S. & Johnson, David C. "Some Thoughts About Problem Solving." Arithmetic Teacher. April 1967. pp. 261 - 262.

Department of Education, State of Hawaii. Mathematics Program Guide. Honolulu, HI: DOE, State of Hawaii. 1978.

Henderson, Kenneth and Pingry, Robert. "Problem Solving in Mathematics." The Learning of Mathematics: Its Theory and Practice. Twenty-first Yearbook of the National Council of Teachers of Mathematics, Washington, D.C.: The National Council of Teachers of Mathematics, 1953. pp. 228 - 270.

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Noller, Ruth B.; Heintz, Ruth E. and Blaeuer David A. Creative Problem Solving in Mathematics, Buffalo: D.O.K. Publishers Inc., 1978.

Wickelgreen, Wayne A. How to Solve Problems. San Francisco: W.H. Freeman, 1974.

UNIT II

Figuring It Out

Introduction

A critical technique of problem solving is that of drawing, labeling and reading diagrams. Diagrams are of great value in problem solving because they help to clarify the problem situation. The "givens" of the problem are available at a glance. Also, diagrams could suggest to the problem solver a possible plan of attack. Solutions could even be estimated for some problems. This is especially true for scale drawings.

Students should be encouraged to draw accurate diagrams. Research has shown that inaccurate diagrams are worse than no diagrams at all.

Learner Objectives: Students will

1. Given a description of a situation, draw and label a representation of it.
2. Given a labeled drawing, write a verbal description of it.
3. Given a labeled drawing, write word problems using the information depicted by the diagram.
4. Solve problems by drawing diagrams or by utilizing the information given in a diagram.
5. Given a diagram of the number line, write word problems using the information depicted.
6. Given a description of a situation, draw and label the number line to represent the situation.
7. Solve problems by drawing and labeling the number line or by using the information given by a number line diagram.
8. Given a description of a situation requiring a scale drawing, draw and label a scale drawing.
9. Given a scale drawing, write word problems using the information depicted.

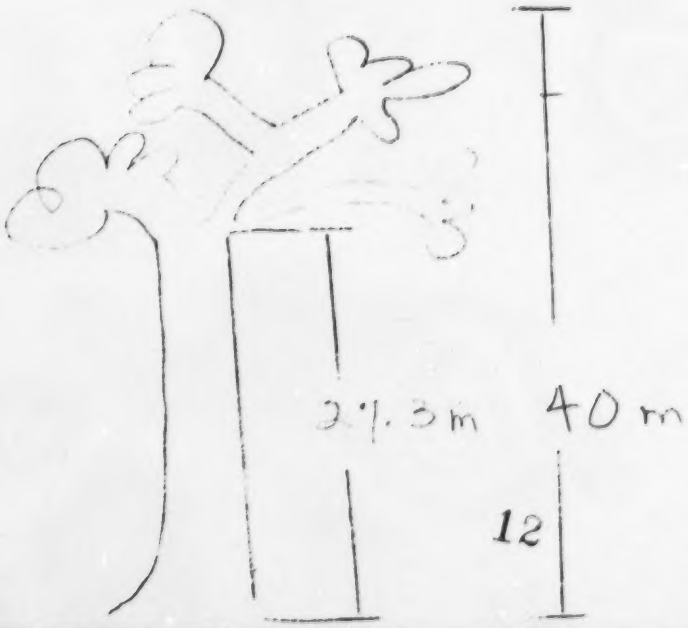
10. Solve problems by using a scale drawing where appropriate or by using the information given by a scale drawing.

Instructional Strategy

Students should be informed of and experience the value of drawing, reading, and labeling diagrams. They will need to be reminded and encouraged frequently to sketch diagrams. The teacher should share with students how he/she proceeds in sketching and labeling diagrams. In addition students should also share how they proceed in sketching and labeling diagrams.

Sample Activities

1. Answer the following questions about the diagram. If the diagram does not show enough information to answer the question, state not shown.
   a. How tall is the tree?
   b. At what height does it begin to branch out?
   c. How high are the branches?
   d. What kind of tree is it?
   e. How many branches does the tree have?
   f. How deep are the roots?



2. The Higashi's Olympic-size swimming pool is 50 meters long and 22.85 meters wide. The average depth is 1.524 meters. The Higashi's plan to put a fence around the pool. They also want room for walking around the pool. Hence they plan to allow 1.5 meters around the pool. How many meters of fencing will they need to buy to enclose the pool?

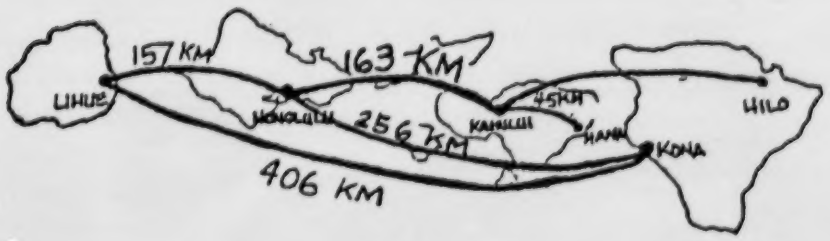
   a. Draw a rectangle to represent the swimming pool.

   b. Label its width and length.

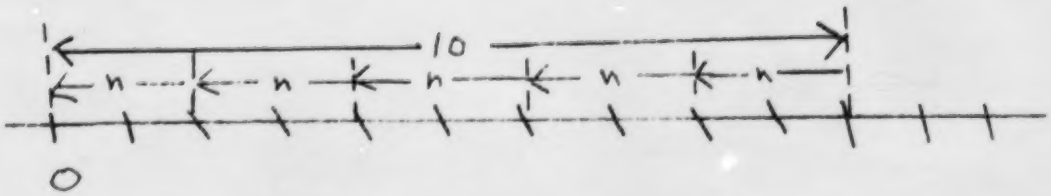
   c. Draw in the walk and label its width.

   d. What does the problem ask you to find? Darken this on your diagram

3. What facts do you see in the diagram? Make up problems using the facts. Ask a friend to solve your problems.



Airline Route Map

4. 

Write a word problem that would be represented by the number-line diagram shown.

Resources

Teacher:

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Sherrill, James. "The effects of different presentations of mathematical word problems upon the achievement of tenth grade students." School Science and Mathematics. April 1973, pp. 277-282.

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Student:

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Spitler, Gail. MP 1 Problem Solving. Chicago: Rand McNally & Co., 1976.

Taylor, Naomi L. Problem Solving. Orange and Olive Levels. New York: William H. Sadlier, Inc., 1975.

UNIT III

Organizing Information

Introduction

Another technique useful for solving problems is that of organizing data or information. This unit will look at ways of organizing information by means of organized listings including the listing, where possible, of all possible outcomes or all possible cases, by using tree diagrams, and by making tables and graphs.

Although the focus of this course is on problem-solving strategies, the content areas of Option X, Level B courses that lend themselves to teaching the technique of organizing data should not be overlooked. These areas include statistics, probability, and counting problems.

The general technique of organizing information may be viewed in terms of these processing activities: labeling, listing, selecting according to criteria, sequencing, transforming, estimating, and summarizing.

The organization of information alone is not sufficient for solving problems. Once organized, the data must be interpreted. Hence this unit will also focus on "reading" the data once it is organized. This reading may be viewed in terms of these processing activities: decoding verbal and non-verbal symbols, inferring, interpolating, extrapolating, relating and equating, comparing, generalizing and predicting.

Learner Objectives: Students will

1. Given a description of a situation, label and classify appropriate outcomes.
2. Solve problems by constructing an organized list or by using information given in such a list.

3. Organize lists by making tree diagrams.
4. Read and interpret tree diagrams.
5. Solve problems by constructing and/or by using the information contained in tree diagrams.
6. Organize information by using tables.
7. Solve problems by constructing and/or by using the information in tables.
8. Organize information by using graphs, including circle graphs, bar graphs, and pictograms.
9. Use graphs to see relationships and to make predictions.
10. Solve problems by organizing information using graphs and/or by interpreting graphs.

Instructional Strategy

The data to be organized may result from students working at the concrete, representational (pictorial), or abstract levels. For example, in the problem: In how many different ways can you make change from fifty cents without using pennies? Students could work with coins, drawings of coins, or names of the coins. There are various instructional strategies that may be incorporated for each level. A discussion of some of these follows:

1. Laboratory-type Lessons. These are lessons that incorporate a work-sheet wherein students work at the concrete level. The sequence of activities for the students is inductive in nature. Students are encouraged to experiment and to make conjectures.
2. Projects. Projects encourage independent investigations. They generally take longer to accomplish than daily assignments. Generally, projects should be assigned to pairs or groups of individuals in order

to shorten the time needed to collect and organize data. Students in the groupings should discuss the mathematics they are studying and share their ideas on how best to organize data. By so doing the students will be teaching and learning from each other.

3. Problem Station. Because of the time that might be needed to gather and organize data, some problems may be presented as problems of the week, or month. These may be placed at a problem station. Physical materials useful for solving problems should be available.

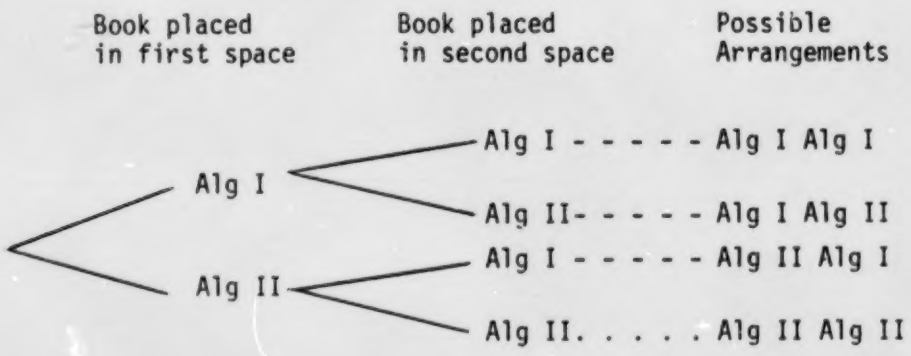
4. Personalized Data. The self as an object of study is always motivating. The average height of boys and girls in the classroom could be studied as a class problem in which every student makes an input. In using personalized data avoid embarrassing any student.

Sample Activities

1. Growing Trees. Prior to the drawing of tree diagrams it may be necessary for students to actually manipulate two books (or other objects) to see what arrangements of books are possible. The tree diagram then becomes a record of the results of their manipulating (arranging) the books. The need to systematically arrange books and then record results should be stressed.

   Problem: The school library has several identical copies of Algebra I and Algebra II. There are spaces for 2 books on the shelf. In how many ways can the shelf be filled with books so that each arrangement looks different?

Solution:


Book placed in first space
Book placed in second space
Possible Arrangements
Alg I
Alg II
Alg I - - - - - Alg I Alg I
Alg II- - - - - Alg I Alg II
Alg I - - - - - Alg II Alg I
Alg II. . . . . Alg II Alg II

2. Using Tables.

Problem: Ana, Babs, and Cindy were the first three to complete the 50 meter swim heat. Using the information below, decide who came in first, second, and third. (Hint: Make a table.)

a. Cindy has never finished first in a 50 meter swim heat.

b Babs always finishes first or second.

c. Ana's sister finished third.

d. Babs is younger than the one who finished second.

Solution:

	First	Second	Third
Ana	X		X
Babs		X	X
Cindy	X	X	

3. Picturing Relationships.

Make a pictograph of the following data. Then answer these questions.

a. How many years did it take for the world's population to double?

b. How many years did it take for the population to double again?

c. How many years did it take for the population to double a third time?

d. Estimate the world population in the year 2000.

WORLD POPULATION

Year	Number of People
1	250 million
1000	276 million
1500	460 million
1600	486 million
1750	696 million
1800	917 million
1850	1090 million
1900	1575 million
1950	2490 million
1965	3280 million

Resources

The following references consist of ideas and activities for use in the mathematics classroom:

Bright, George W. "Using Tables To Solve Some Geometry Problems." Arithmetic Teacher. May 1978. pp. 39-43.

Columbus Junior High ESEA Title III. Mathematics For Problem Solving. Columbus, Montana: Columbus Junior High, 1975. pp. 13-15.

Greenes, Carole; Gergory, John; Seymour, Dale. Successful Problem Solving Techniques. Palo Alto, CA: Creative Publications, 1977. pp. 10-11, 38-45.

Greenes, Carole; Spungin, Rika; Dombrowski, Justine. Problem-mathics: Mathematics Challenge Problems With Solution Strategies. Palo Alto, CA: Creative Publications, 1977. pp. 7, 16-59, 30 & 91.

Iowa Problem-Solving Project. "Problem Solving Using Tables." Cedar Falls, I: Iowa Department of Public Instruction, 1978.

National Council of Teachers of Mathematics. Booklet 16: Collecting, Organizing, and Interpreting Data in Topics in Mathematics Series, Washington, D.C.: National Council of Teachers of Mathematics, 1969.

National Council of Teachers of Mathematics. Arrangements and Selections. Unit Five of Experiences in Mathematical Discovery. Washington, D.C.: National Council of Teachers of Mathematics, 1966.

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Mathematics Resource Project. Statistics and Information Organization. Preliminary Edition. Palo Alto, CA: Creative Publications, 1977. pp. 341-352.

Vos, Kenneth E. Problem Solving Organizers. 990 68th Ave. N.E. Minneapolis: Kenneth E. Vos, 1975. pp. 2, 6, 7 and Instruction 310, 320, 330, 340, 510, 520, 530, 540.

Whitman, Nancy C. Experiments For Secondary School Geometry. Honolulu, HI: the author, 1972.

Wirtz, Robert W. Drill and Practice at the Problem Solving Level. Washington, D.C.: Curriculum Development Associates, Inc., 1974.

UNIT IV

Using Patterns

Introduction

The search for patterns is an ongoing human endeavor. Knowledge of patterns helps us plan our daily activities. For example, knowledge that the mail generally comes by 2:00 p.m. daily encourages us to deposit mail by 2:00 p.m. if we want the mail to go out on that day. Scientists, in attempting to understand the universe, make observations about the universe; that is, they are searching for regularities or patterns that relate various aspects of the universe. Based on these regularities scientists are able to make predictions about future events. Mathematicians also study their subject matter in search of regularities. In fact some mathematicians consider mathematics to be the study of patterns. It is no wonder then that the strategy of looking for patterns is one used by mathematicians to solve problems.

To be able to use patterns to solve problems, students will need to have experiences in searching for and identifying patterns. These patterns should be taken from several areas of mathematics to make students aware of the pervasiveness of patterns in mathematics. As much as possible the content areas should also reflect the Level B learner objectives. Examples of activities to engage students in pattern search and recognition in the areas of numbers, geometry, measurement are given under Sample Activities.

At times the strategy of searching for patterns is closely tied in to that of organizing data for problem solving. For example, with certain problems in mathematics students will need to first put known or available mathematics data into a systematic arrangement before proceeding to a possible solution.

Learner Objectives: Students will

1. Put known or available mathematics data into a systematic arrangement.
2. Search for and identify patterns.
3. Use patterns to make predictions.
4. Use patterns to form generalizations.
5. Use patterns to solve problems.
6. Use patterns to extend problems.
7. Use patterns to generate questions as a basis for investigation.
8. Recognize that the study of patterns is common to many human endeavors.

Instructional Strategy

Most students will enjoy searching for and identifying patterns. The teacher nonetheless will need to be sensitive to the psychology of the learning situation. He/she will need to decide whether to have the learning activity occur as an individual, a small group, or a whole class endeavor. Whatever mode is used, the teacher must be prepared to accept all patterns that are indicated, not only the ones the teacher recognizes. All students should have some success in recognizing patterns; hence, the teacher should encourage students to "write down" or "whisper to the teacher" their discovery if the students cannot "keep still" with their discovery. The teacher thus makes it possible for the other students in the class to "mull over" the materials presented. Students should all be given ample time to think about the situations presented. Too frequently students are asked to give almost instantaneous answers.

In having students generalize from specific cases, the teacher should provide at least 5-6 instances. This generally will assist students in

seeing a pattern. Once students make a generalization they should be encouraged to verify or check it by using additional cases.

Since pattern recognition and identification activities might occur as an individual, group, or class activity, the general strategies of guided discovery, individualized instruction, and laboratory activities are all possible strategies the teacher might employ.

The teacher might introduce pattern recognition activities for a variety of reasons. Some of these, in addition to that of teaching a strategy to aid in solving problems, are: to introduce a concept or generalization, to provide drill, and to provide challenge problems.

Sample Activities

1. This activity is from the area of geometry and measurement. It is written as a laboratory-type lesson. The purpose of the lesson is to provide experiences in pattern recognition and in making conjectures. It also provides students with experiences in applying their knowledge of terms such as right triangle, median, hypotenuse, and right triangles. The activity might be used as individual, small group, or class activity.

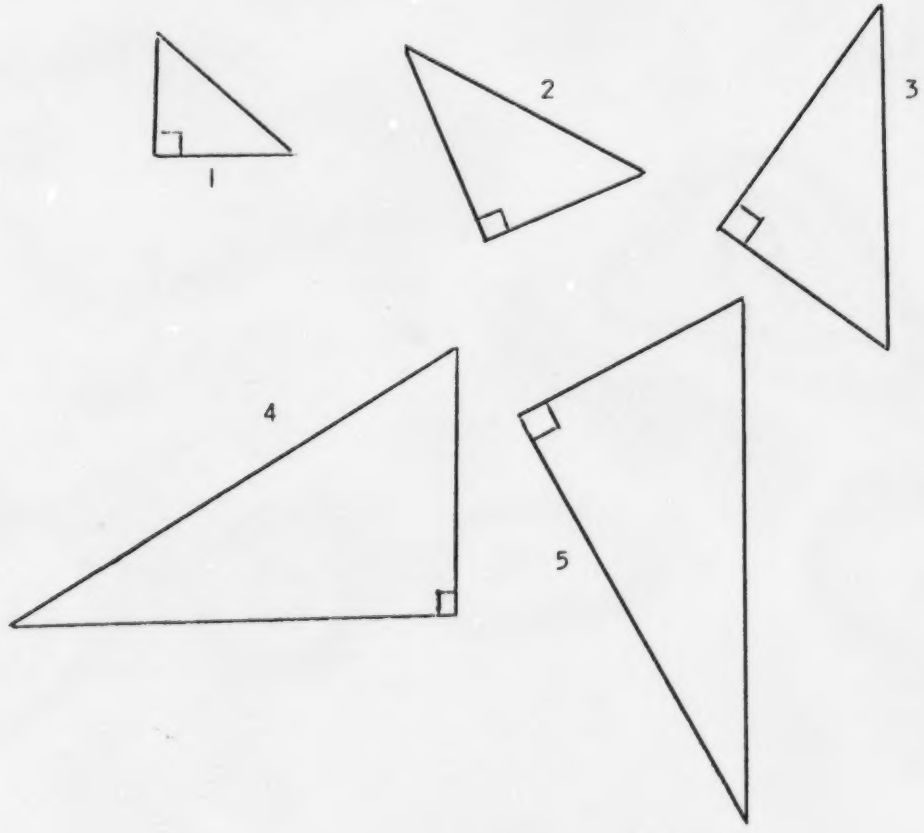
Sample Activities (continued):

IN A RIGHT TRIANGLE, HOW IS THE MEDIAN FROM THE RIGHT ANGLE RELATED TO THE HYPOTENUSE? *

MATERIALS: ruler, pencil, paper

PROCEDURE:

1. Look at the following examples of right triangles of different sizes.



2. For each right triangle, construct the median from the right angle to the hypotenuse.
3. Measure each median you construct, and measure its corresponding hypotenuse, place your measurements in the table provided below.

*From Geometry in Secondary School by Nancy C. Whitman, c. 1972.

Sample Activities (continued):

RESULTS:

Triangle	Hypotenuse Measure	Median Measure

1. Study your table. What patterns do you recognize?
2. What pattern or connection do you recognize about the measures of the hypotenuse.

CONCLUSIONS:

1. Make up a rule for <u>all</u> right triangles about the connection between the hypotenuse and the median from the right angle.
2. See if your rule works by drawing another right triangle of a different size than the ones drawn, and measure the hypotenuse and the median to the hypotenuse.

Sample Activities (continued):

2. This activity is from the area of numbers and operations. It deals with exponents. The purpose of this lesson is to introduce the idea of exponents and the related idea of bases. In addition to providing experiences in pattern recognition, it can provide drill on number facts.

LOGARITHMS*

A bacterium reproduces at a terrific rate. In five hours time, one bacterium can divide into 32,768 bacteria! In 20 minutes, a bacterium will divide into two bacteria; in 40 minutes these two bacteria will divide into four bacteria; in 60 minutes these four bacteria will divide into eight bacteria; and so on. Thus, this can be seen as:

Number of 20 mins.	0	1	2	3	4	5	6	7	8	9	10
Number of bacteria	1	2	4	8	16	32	64	128	256	512	1024

How many bacteria will exist in 120 minutes? in 140 minutes? in 200 minutes?

Exercise 1

1. What pattern do you recognize in the first sequence of numbers?
2. What pattern do you recognize in the second sequence of numbers?
3. If any, what pattern exists between the two sequences?
4. Continue the sequences until each sequence has 15 numbers. Save your results to use later.

---

*From Computation III, Honolulu: Curriculum Research & Development Group, 1978.

Sample Activities (continued)

*3. This activity assists the student in organizing data in order to aid him/her in seeing patterns.

The Class of '81 is having a picnic. People arrived in an interesting manner. The advisor arrived first alone. Three people arrived in the second vehicle, 5 people in the third, 7 people in the fourth and so on. The table shows this information. Fill in the boxes provided.

HINTS:
(1) Do you notice a pattern in the second column?
(2) Do you notice a pattern in the third column?
(3) What relationship is there between the numbers in column 1 and column 3?
(4) What relationship is there between the numbers in column 1 and column 2?

Vehicle	Number of People in Vehicle	Total Number at Picnic
1	1	1
2	3	4
3	5	9
4	7	☐
5	☐	☐
6	☐	☐
7	☐	☐
.	.	.
.	.	.
.	.	.
☐	☐	100
.	.	.
.	.	.
.	.	.
25	☐	☐

4. This activity attempts to provide experiences by which student might attain learner objectives 1-5 previously listed.

Leilani is having a baby luau. She expects 1,000 guests. At 3:00 p.m. the first knock and guests arrive. On each successive knock a group arrives that has two more guests than the group that arrived on the previous knock.

a. How many guests arrive on the seventh knock?

b. How many guests would have arrived on the 15th knock? on the 20th knock?

*From Applications A_1 draft: Curriculum Research and Development Group, 1979.

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UNIT V

Simplifying Problems

(or Getting Down to Cases)

Introduction

The strategy of simplifying a problem in order to solve it has several facets. What follows is a discussion of some of these.

1. Special Cases. This technique is frequently seen in conjunction with that of using patterns. The mathematical content that many times lends itself to this approach is that of sequences. For example, to solve the problem:

   Find the sum: $1 + 2 + 3 + \ldots + 50 + 49 + 48 + \ldots + 1$

   one could first find the sum of special cases as follows:

   $1 + 2 + 1 = 4$

   $1 + 2 + 3 + 2 + 1 = 9$

   $1 + 2 + 3 + 4 + 3 + 2 + 1 = 16$

   $1 + 2 + 3 + 4 + 5 + 4 + 3 + 2 + 1 = 25$

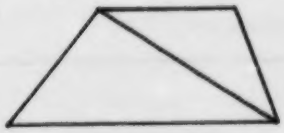
   From these special cases one might note that the sum is always a square number. In fact, it is the square of the middlemost number. Hence the solution to the original problem is 50^2 or 2500. Note that the special cases worked with start from the simplest and note that along with the technique of looking at special cases one also utilized the method of looking for patterns.

2. Decomposition. This technique breaks the problem down into parts of the total problem. For example, to solve the problem:

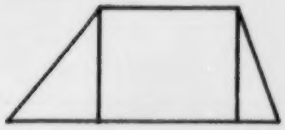
   Find the area of a trapezoid.

   one could break it down into parts that are easier to solve and

which together make up the whole. Possible approaches might be to decompose the trapezoid into two triangles



or to decompose it into two triangles and a rectangle.



3. Analogies. A simpler problem could be an analogous one. Hence the technique of using analogies to solve problems. These dimensional problems might be solved by studying its two dimensional counterpart. For example, the problem:

   Which rectangular prism with a given surface area has the greatest volume?

   might be solved by solving the analogous problem:

   Which rectangle with a given perimeter has the greatest area?

4. Exhaustion. In using this technique one has to identify all possible situations or cases. For example, for the problem:

   If the product of two integers is $^-24$ and their sum is $^-2$, what are the two integers?

   one could list all pairs of integers whose product is $^-24$.

Product = $^-24$	Sum
24, $^-1$	23
$^-24$, 1	$^-23$
12, $^-2$	10
$^-12$, 2	$^-10$
8, $^-3$	5
$^-8$, 3	$^-5$
6, $^-4$	2
$^-6$, 4	$^-2$

Since the list contains all the possible products, one can solve the problem by selecting the desired sum.

This technique is obviously restricted to problems wherein all possible situations can be identified. However, it is a very useful technique where it can be applied. Note that this technique also makes use of the technique of organizing information. As you might have noticed, the mathematical copies of permutation and combination which is a rich source of problems is also a good source for problems where all possible cases can be identified.

Learner Objectives: Students will

1. Identify and solve special cases of a given problem.
2. Solve problems by identifying and solving special cases of the problems.
3. Given a problem that might be decomposed into simpler problems, devise ways to decompose the problem.
4. Solve problems by first decomposing them and then solving the simpler subproblems.
5. Identify analogous problems.
6. Solve problems by utilizing the knowledge obtained by studying and/or solving an analogous problem.
7. Given a problem wherein all possible situations may be identified, list all the cases.
8. Solve problems by utilizing the information given in an exhaustive list of possible situations.
9. Recognize, describe, and discuss the general technique of simplifying the problem.

Instructional Strategy

The experiences that provide students opportunities to use the heuristic of solving simpler problems may be quite varied. Class discussions, laboratory lessons, small group work, and demonstration/lecture by the teacher may all be used. With certain decomposition problems, the use of geoboards and graph paper by students should be encouraged. In some cases the students should be encouraged to cut up and rearrange parts of plane figures. The use of overhead transparencies by the teacher would help to clarify certain decompositions--especially the geometric ones. This is especially true if he/she wants to demonstrate movement of some sort.

In using analogies to some problems, students should be encouraged to work with physical models and drawings where appropriate. The language of analogies should be stressed. Many students find the use of analogies difficult.

It is very necessary for the teacher to model the various techniques discussed here. Not only should he/she model the various techniques but an explanation of the thinking used in proceeding with the use of the techniques to solve problems should be shared with the student. Furthermore, as students begin to use the techniques they in turn should share their thinking with the rest of the class. In this way, students will see the variety of ways in which problems are attacked and solved.

Students should realize that a variety of problem-solving techniques are frequently used to solve problems. In fact, students may be using techniques that the teacher has not yet introduced to the class. Nonetheless, all techniques suggested should be seriously considered. In fact, students should be praised where they use techniques of their own.

Whenever possible, a variety of mathematical content should be used

when providing students with problems to solve. Avoid the possibility of your students associating a particular content area with a particular problem-solving approach. For example, if all decomposition problems are geometric in nature, students may fail to use the technique with non-geometric problems. Hence you may have inadvertently helped students develop a mind set.

Sample Activities

1. Decomposition. The following laboratory type lesson engages students with the physical act of decomposition. Numerous activities and demonstrations of this type should sensitize students to the technique.

EXPERIMENT 13-A*

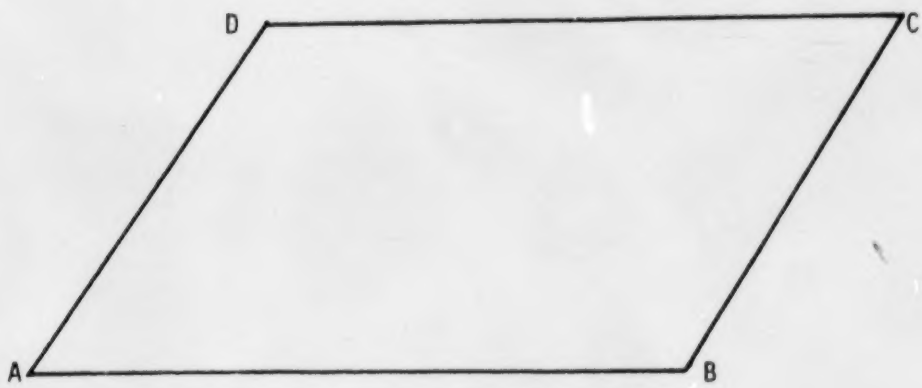
WHAT IS THE AREA OF ANY PARALLELOGRAM?

To answer the question above, we will use our previous knowledge of the area of a rectangle. Here as elsewhere in mathematics, we extend our knowledge by building on our previous knowledge.

MATERIALS: scissors, pencil, tracing paper

PROCEDURE:

1. Trace the following parallelogram.



* Taken from Geometry in Secondary School, by Nancy C. Whitman, c. 1972.

2. Cut out the parallelogram.

3. Fold the parallelogram as in Figure 1 below so that $\overline{DE}$ is perpendicular to $\overline{EB}$.

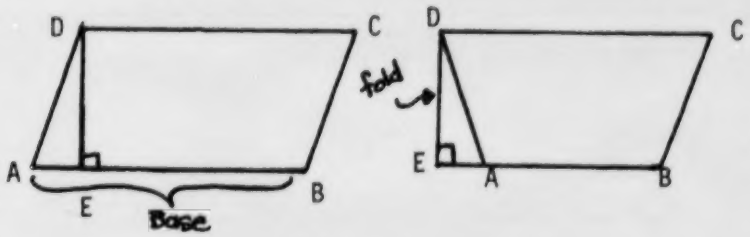


Figure 1

4. Cut along the fold.

5. Form a rectangle with the two cut pieces.

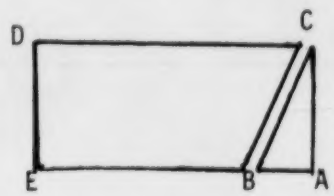


Figure 2

RESULTS AND CONCLUSIONS:

1. Is the area of the parallelogram equal to the area of the rectangle? _____.

2. Since the formula for the area of a rectangle is:

   Area = Base x Altitude

   what would be the formula for the area of a parallelogram?

   A = _____

2. Special Cases.

a. What is the sum of the angles of a 20-sided polygon?

The simpler questions are: What is the sum of a 3-sided polygon? of a 4-sided polygon? of a 5-sided polygon? Depending on the students background, answers to the three questions may be obtained in a variety of ways. These include recall, conjecture based on knowledge of angle

sizes, experimentation, and deduction. Once the questions are answered, students should be encouraged to organize the information in a systematic fashion. It could be organized as follows:

Kind of Polygon	Sum of Angles
3-sided	180^{o}
4-sided	360^{o}
5-sided	540^{o}

Students should also be encouraged to look for relationships within the organized information. Questions such as: What do you notice about the sum of the angles as the number of sides increases? By how much does each sum increase? Some students might be helped if they draw pictures of the polygons. The teacher should also encourage this.

b. Have students simplify problems by having them use "easy" numbers in place of "difficult" numbers in the problems. This technique makes it easier for students to see the structure of the problems.

> It costs about \$0.075 to travel one kilometer on an airplane. How much will a 3246 km plane trip cost?

The problem may be simplified by replacing \$0.075 by \$0.10 and 3246 by 3000. The easier numbers should be chosen by the students.

3. Exhaustive. Have students find the digit represented by each letter in the following division problem where each letter stands for a different digit. The need to systematize one's approach might be pointed out to the students.

```
          X T X T
T D X ) E D E I T X
        E S X
          S S I
          T D X
            E X T
            E S X
              S N X
              T D X
              T S I
```

In this example, in addition to exhausting possibilities, students use other techniques such as trial and error and verification to name a couple.

4. Analogies.

   a. Provide students with a list of plane figures. Ask them to name analogous figures in space. Discuss how the plane and solid figures are alike and different.

   b. Have students make statements about plane figures that are true. Then have them name statements analogous to that named that may be true in space. These statements could then be verified through experimentation with physical objects.

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UNIT VI
Solving Word Problems

Introduction

The solving of word problems remains a difficult task for many students. A range of reasons may account for the difficulty. However, a reason given by many educators is that students lack the necessary reading skills. Another reason why students may have difficulty in solving word problems is that they lack the problem-solving skills discussed in the previous units. In solving word problems students are called upon to utilize both reading and general problem-solving skills. The bridging of these two skills is demanding (of itself) and may account for students' difficulty.

This unit attempts to assist students in reading mathematics and in applying the previously studied problem-solving skills. Of the previously studied problem-solving skills those of drawing, labeling, and reading diagrams; making inferences from the given, the goal, and between the goal and the given; and making estimates of solutions to problems will be especially helpful in solving word problems.

In solving word problems it is assumed that students "see" and "hear" the language used. For example, when the teacher says "ratio", students should hear "ratio" and not "radio". And when presented with the word "tenths", they see "tenths" and not "tens". Second, it is assumed that students understand the key vocabulary used in the problem. For example, in the problem "Ms. Singer plans to cane the seat of her favorite chair. The length of it is 22 centimeters and its width is 20 centimeters. What is the area that must be caned?" It is assumed that students understand the words "cane", "length", "width", "centimeters", and "area". Furthermore, it is assumed that students understand the relationship of the terms used. In the previous example it is

assumed that students know the relationship of width, length, and area. And lastly, it is assumed that students will apply their knowledge of problem-solving skills. When any of these assumptions are false, students will experience difficulty in solving a word problem.

Learner Objectives: Students will

1. Recognize, state, and describe key words in a word problem.
2. Express a word problem in their own words.
3. Use a suitable pictorial or graphic representation to aid in the solution of word problems.
4. Recognize relevant information in a word problem:
   a. Identify the "givens" and the "unknown".
   b. Decide what information is needed to solve the problem.
5. Estimate answers to a word problem.
   a. Use "easier" numbers.
   b. Use patterns, trends.
   c. Guess and refine guesses.
6. Express a verbal phrase or clause symbolically.
7. Write a mathematical sentence to aid in the solution of a word problem.
   a. Decide which operation(s) to use to solve a problem.
   b. Use appropriate formulas, e.g., $I = PRT$, $U = \ell wh$, $p = 2\ell + 2w$.
8. Make up word problems given the operations.
9. Determine whether their solution fits the context of the problem.
10. Realize that there is usually more than one way to organize and solve a mathematics word problem.

Instructional Strategy

1. Assumptions. The assumptions regarding students' knowledge to solve word problems need to be verified. Techniques such as having students read aloud, write, or explain relationships could be used. Whenever an assumption is false, adequate experience must be provided students to correct the void.

2. Content Selection. Only the most important skills, words, concepts, and relationships should be tackled. An identification of these must be made before the start of instruction. Students should be informed of your selections.

3. Communication. Provide numerous experiences wherein students communicate orally and in writing. The use of buzz groups to explain and justify statements affords a student who reports to the class the backing of a group. Reading aloud by teacher and students allows students to "see" and "hear" mathematics. The use of "working in pairs" encourages communication. Many ideas on how to solve a problem are obtained by listening to how others solved the problem. Hence, the sharing of different solutions to a problem is encouraged.

4. Range of Experiences. In developing vocabulary with students, provide concrete to abstract experiences. Real life examples are the most concrete, with a formal definition being the most abstract. Between these two extremes are 1) simulated examples such as figures drawn on a chalkboard, cut from paper material, or painted out in pictures and diagrams; and 2) the listing of characteristics. For example, several things might be listed about a square. It has four sides. All of its sides are equal. It has four right angles.

Take advantage of the many opportunities to bombard students with words either orally or visually. Mathematical word games and bulletin board displays are helpful.

5. Problem Solving Guides. Provide students with a written guide wherein the general stages of problem solving are applied to a specific word problem. The guide should pose specific questions that simulate the process of reading word problems and it should actively involve the student at each stage of the process. The teacher should solve the problem and be aware of the process used before constructing a guide.

6. Creating Word Problems. Provide students opportunity to create and solve each other's problems. Also the identification and solution of problems from magazines, newspapers, and books will add variety.

Sample Activities

1. A problem-solving guide. Item A relates to what is wanted, item B to what is given, item C to the relationship of what is given to what is wanted, and item D to the examination of the solution.

Jo has $1.75. She wants to buy four greeting cards at 35¢ each. She also wants to buy bubble gum at 6¢ each. If she buys the greeting cards, how many bubble gums can she buy with the money left?

A. Check one answer.

In this problem you are trying to find

_____ the price of the greeting cards Jo wants to buy

_____ the money Jo has to begin with

_____ the cost of bubble gum

_____ the number of 6¢ bubble gum Jo can buy.

B. Write in the correct amount. Refer back to the story.

_____ the amount of money Jo had to begin with

_____ the amount she paid for 4 greeting cards at 35¢ each

_____ the amount of money she would have left after buying 4 greeting cards at 35¢ each.

C. Write a sentence relating the cost of a bubble gum, the amount of money Jo had left after buying the greeting cards, and the number of bubble gums Jo can buy.

D. Complete this problem using the work done so far. Be sure to label your answer.

2. A crossword puzzle to develop vocabulary.

1. An instrument used to construct circles and arcs of circles.
2. The middle of a circle.
3. A line segment that passes through the center of the circle and has its endpoints on the circle.
4. A set of points the same distance from a given point.
5. A part of a circle.
6. A line segment with one endpoint on a circle and the other at the center.
7. An instrument for measuring angles.
8. The distance around a circle.
9. An arc whose length is half a circle.
10. Angles whose sides are radii of a circle and whose common vertex is the center of the circle.

The following problems are intended to focus students' attention on relevant information needed to solve word problems:

Problems*

Directions: Read each problem on the left. Answer the question to its right.

1. Joe, Maria, Jim, Lisa, Bob, and Mary shared expenses for a party. The food cost $30. How much was each one's share?	How many people helped to pay for the party? a. 1 b. 6 c. 10 d. 30

*From a draft of Applications B_1, being developed by the Curriculum Research and Development Group, College of Education, University of Hawaii.

2. Jack earns $3.55 an hour. He works 36.5 hours a week. How much does he earn in one year?	What information is needed to solve this problem? a. How much Jack earns in one month b. How many weeks in one month c. How many weeks in one year d. All necessary information is given
3. Fran took four math tests this quarter. Her scores were 80, 69, 92, and 73. What was her average for the quarter?	What does the question ask you to find? a. Fran's total points scored b. Fran's highest test score c. Fran's grade for the quarter d. Fran's average on four tests

4. Place a check mark (✓) next to each question that can be answered from the information given.

   Paul pedals a pedicab in Waikiki. He charges a $1.50 base fee and $1.25 for each mile travelled.

   ____ How much will a 5-mile ride cost?

   ____ If you want to spend $4 on a ride, how many miles could you ride?

   ____ Could you find a cheaper rate from someone else?

   ____ Is it fair to charge the same rate regardless of whether one or two people ride the pedicab?

   ____ If Paul picks up 10 rides for a total of 30 miles in one night, how much did he make?

5. Write a question that can be answered using the following facts:

   Sandy won the election for student body president. She received 583 votes. Mike came in second with 416 votes and Jill had 58 votes less than Mike.

   Question: _____

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UNIT VII

LOOKING BACK

Introduction

This strategy is a difficult one for many students. Even the best mathematics students overlook this important strategy. They fail to realize that frequently there is more to be learned in looking back at the solution process than the solution itself. By looking back at the solution process students will be learning procedures that are applicable to numerous problems, not only to the one just solved. Also, students will see that mathematics does not consist of unrelated problems but rather will see that the problems in mathematics are related. "By looking back at the completed solution, by reconsidering and reexamining the result and the path that led to it, they could consolidate their knowledge and develop their ability to solve problems."*

The strategy of looking back may be perceived as: 1) checking results and procedures, 2) applying procedures and results, 3) deriving results differently, 4) deriving other results, and 5) generating additional related problems. What follows is a discussion of each of these.

1. Checking Results and Procedures.

   Checking the result of a problem does not consist of asking the teacher if the answer is correct nor of looking up the answer in an answer key. Some of the behaviors that consists of checking a problems are:

   a. determining if all the conditions of the problem have been met.

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*From pages 14-15 of How To Solve It by George Polya.

For example, in the problem:

> Find 2 two-digit numbers with the same digits such that their sum equals 88 and their difference equals 36.

A check would consist of determining if: 1) the answer has 2 two-digit numbers, 2) the numbers have the same digits, 3) the sum of the numbers is 88, 4) the difference of the numbers is 36.

b. relating the solution to known or observable facts.

For example, in determining the amount of discount on an item, the answer should be within reason. An item discounted at 25% should not cost more than the cost before the discount. Also the average weight of high school males should not be 12.53 pounds. Both solutions are not within known facts.

c. changing the order or sequence in which the problem is executed.

For example, in the problem:

Solve for x: $3(2x + 5x) + 6 = 0$

if students first added 2x and 5x and then multiplied 7x by 3, the order could then be changed to check the result. Hence, students could first multiply 2x by 3 and 5x by 3 and then add 6x and 15x. Results of column addition problems are also frequently checked by reversing the order in which the calculations are done.

d. applying the result to a specialized case. This check is especially helpful if the solution consists of "variables" instead of numbers. For example, to determine if the solution

of the following problem:

$$\text{Simplify: } \frac{(a^2b^3)^2 + a^2c^3}{a^2}$$

is correct one could replace each of the "letters" with specific numbers and carry out the required computation.

2. Applying Procedures and/or Results.

After a problem is solved, it is very instructional to review the procedures used. The procedures should be clear in the student's mind as is the solution.

To help clinch the procedures used and result obtained, the procedure and/or result should be applied to other similar problems. The result in particular should be applied to a physical or concrete situation.

In addition, the procedure could be applied to a generalized problem. For example, if the problem:

Find the sum: $1 + 2 + 3 + 4 + \ldots + 47 + 48 + 49 + 50$

is solved as follows:

$$\begin{array}{r} 1 + 2 + 3 + 4 + \ldots + 47 + 48 + 49 + 50 \\ \underline{50 + 49 + 48 + 47 + \ldots + 4 + 3 + 2 + 1} \end{array}$$

therefore:

$$\text{Sum} = \frac{50\ (51)}{2} = \frac{2550}{2} = 1275$$

the question should be raised as to whether the technique or procedure would work no matter how many terms are used. In order for students to answer this question it must be crystal clear to the students just what procedure was used.

3. Deriving Results Differently.

   The strategy of looking back at the solution procedure and obtaining the same result by using another procedure not only focuses on procedures but also develops in the student greater confidence in the result obtained.

4. Deriving Other Answers.

   In searching for alternative answers one might review the procedures used and study others that may be utilized. Further the procedure first used might be refined, modified, or completely changed. In the problem:

   > Find two numbers such that twice the first number and three times the second are equal to 17.

   students might first guess and answer, then they might decide to systematically list possible solutions. For additional similar problems, students might decide to always use the latter method.

5. Generating Additional Related Problems.

   Another technique that compels students to focus on the solution process is that of having students generate related problems. These problems may be a specialized case of the original problem, a generalization of the original problem, or a problem analogous to the original problem. This looking back technique is useful in helping students to solve word problems.

Learner Objectives: Students will

having solved a given problem

1. Determine if all the conditions of the problems have been met.
2. Relate the problem solution to known or obserable facts.

3. Change if appropriate, the order or sequence in which the problem is solved.
4. Apply if appropriate, the problem solution to a specialized case of the problem.
5. Apply the problem solution to other problems.
6. Apply the problem solution to a physical or concrete situation.
7. Apply if appropriate, the problem solution to a generalized problem.
8. Obtain if appropriate, the same solution using another approach or procedure.
9. Obtain if appropriate, another problem solution.
10. Create problems related to the one solved.

Instructional Strategy

The teacher must communicate to the students the importance of the looking back strategy by their behavior. That is to say, the teacher must model the various behaviors expected of students identified under the learner objectives. Otherwise, students may not consider those behaviors as important as obtaining correct answers. In addition, the teacher should reward correct looking back behaviors. In fact, if looking back is considered as being more important, then more recognition should go to exhibiting those behaviors than to the correct response. To encourage the looking back behaviors, the teacher should provide sufficient time for students to work on problems. Too frequently students have no additional time to reflect on the solution process. Another way that the teacher might encourage the use of the looking back strategy is to frequently ask questions of students that suggest a

need to look back at the solution and the solution process. Questions such as "What does the result tell you?", "Is there another solution?", and "What is the answer to the question?" encourage students to utilize the looking back strategy. Teacher-led discussion wherein students share their solutions and/or processes also focuses attention on the solution process. Such discussion provides the solver of the problem an opportunity to look back at his/her solution. On the other hand the listener will gain insights into different solution processes.

The teacher can help students use the looking back strategy on specific problems by a variety of means. Some of them include:

1. Having students outline the procedures they used;
2. Having students identify and explain the key elements of their problems;
3. Having students use a problem-solving checklist provided by the teacher to determine if the appropriate looking back techniques have been tried, and having students use a problem-solving guide provided by the teacher wherein the appropriate looking back strategies are attended to.

Examples of both a problem-solving guide and a problem-solving checklist to assist students in using the looking back techniques are provided in the next sections.

Sample Activities

1. Looking Back Checklist.

   Provide each student with a checklist similar to the one given below and have them use it after they have solved each problem.

   CHECKLIST

   Answer the following questions after you solve each problem.

   1. Have you answered the question?
   2. Does the label make sense for the question?

3. Is your answer close to your estimate?

4. Can you think of another method now that you have solved the problem?

5. Will you recognize the key elements when you see a problem like this again?

2. Generating Problems.

Have students create problems that are analogous to and problems that are extensions of the following problems after they have solved them.

a. Jim works in an ice cream parlor. Ice cream cones cost 25¢. In how many ways can Jim be paid exactly 25¢ for an ice cream cone?

b. An escalator in a large store is 15m long. It makes an angle of 30 with the horizontal. What is the distance between the two floors?

3. Problem-solving Guide.

Provide students with a problem-solving guide for selected problems.

The following guide is for the problem:

A girl is 6 years older than her brother. The sum of their ages is 20. What are their ages?

PROBLEM-SOLVING GUIDE

a. What are you to find?

b. Who is older? the boy or the girl?

c. How are the ages of the boy and girl related?

d. Write a mathematical sentence expressing the relationship stated for (c) above.

e. Do your answers add up to 20?

f. Do your answers agree with (b) above ?

g. Write a problem similar to this one. Solve it.

Note that in this guide, students are encouraged to check whether the conditions of the problem have been met and to create a problem similar to the problem given.

4. Stating Conditions.

Have students list the conditions of problems, solve the problems, and then check to see if all the conditions of the problems are met.

Problem-Solving Worksheet

For each of the following problems a) list the conditions of the problem, b) solve the problem, and c) check to see if all the conditions of the problem are met.

a. In how many different ways can you make change for fifty cents without using pennies?

b. One number is five times as large as another. Their sum is 72. How large is each number?

c. Dave has 3-1/2 times as much money as Lani. When Dave gives Lani 25 cents, they will have the same amount. How much will each have?

d. In a triangle, one angle is 12^{o} larger than another. The third angle is 6^{o} less than the sum of the other two. Find the measurement of each angle.

Resources

Geier, A. E. & Lamm, N. Mathematics and Your Career. New York: AMSCO School Publications, 1978. See chapter 10 on problem solving. Estimating solutions are stressed.

Hiatt, Arthur A. "Problem Solving in Geometry." Mathematics Teacher. November 1972, pp. 595-600.

Iowa Problem Solving Project. "Problem Solving Using Guesses." Cedar Falls, Iowa: Iowa Department of Public Instruction, 1978.

Jacobs, Russel F. Problem Solving with the Calculator. Phoenix: Jacobs Publishing Company, Inc., 1977.

Morris, Janet P. "Problem Solving with Calculators." Arithmetic Teacher, April 1978, pp. 24-26.

Palagi, George H. "Polya's Triangular Array Problems." Mathematics Teacher, November 1976, pp. 564-566.

Resources (cont'd)

Polya, George. How to Solve It. Second Edition. Princeton, N.J.: Princeton University Press, 1971. See especially pages 14-19.

Pottenger, M. J. and Leth, L. "Problem Solving." Arithmetic Teacher, January 1969, pp. 21-24.

APPENDIX A

PROBLEMS TO SOLVE

In the body of this Guide various strategies and examples were discussed in solving problems. This appendix provides a list of problems with no mention of possible strategies to use. Instead the teacher is encouraged to solve problems in whatever way he/she deems best, but note the strategies you do use and share them with the students.

1. Cynthia and Lorraine have a 24-ounce bottle of soda. There are no markings on the bottle. They each want 12 ounces. They have 3 other bottles. One holds 5 ounces, another holds 11 ounces, and the third holds 13 ounces. How can they pour the soda so that they each get 12 ounces? What is the least number of pourings?

2. Find the length of a diagonal in a 4 x 4 cube.

3. A circle of radius 15 intersects another circle of radius 20 at right angles. What is the difference of the areas of the non-overlapping portions?

4. There are eight people at a party. If each person shakes hands with the other guests, how many handshakes will there be?

5. How many different squares are there in a 10 x 10 grid?

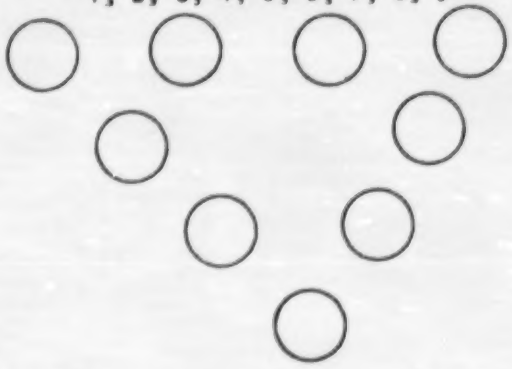
6. Five kids wish to watch Mele Kahea, the baby elephant at the zoo. In how many different arrangements can they line up across the front of the cage?

7. Jim has a collection of nickels and dimes that totals $2.80. He has two more nickels than dimes. How many nickels does he have?

8. Larry raises peacocks and turtles. The animals have a total of 170 feet and 60 heads. How many peacocks and turtles does Larry have?

9. Using each of the following numbers only once, show how you can make each side of the triangle add up to 17. Is this the only way?

1, 2, 3, 4, 5, 6, 7, 8, 9



10. If Ann can paint a room in 4 hours and her friend Dan can paint the same room in 2 hours, how long will it take them to paint the room together?

APPENDIX B
ADDITIONAL RESOURCES

The 1980 Yearbook of the National Council of Teachers of Mathematics is devoted to problem solving. The yearbook will be made available at the April 1980 Annual Conference of the Council.

The following journals are excellent sources of ideas for problem solving:

The Arithmetic Teacher, National Council of Teachers of Mathematics, 1906 Association Drive, Reston, VA 22091

The Mathematics Teacher, National Council of Teachers of Mathematics, 1906 Association Drive, Reston, VA 22091

Mathematics Teaching, The Association of Teachers of Mathematics, Market Street Chambers, Nelson. Lancashire, BB9 7LN, England.

Problem Solving. A monthly newsletter devoted to problem-solving and instruction. It keeps the reader up-to-date on material presented at conferences, as well as on current books and papers, and it reports on programs that are developing methods of teaching problem solving, from elementary through university level, as well as those taught in corporations. Address: Franklin Institute Press, 20th and Race Streets, Box 2266, Philadelphia, PA 19103.

Problem Solving Kit by Wallace P. Judd. Published by Science Research Associates, Palo Alto, Califiornia. The following problem solving strategies are developed in this kit: making a model of the problem, identifying needed additional information, breaking down a problem into parts, simplifying a problem, and researching information. The kit includes a Using Your Calculator workbook, guide, answer key, a problem card answer booklet, 144 problem cards, student record folders, class progress record, and spirit masters of tests. Cost: $99.50.

Problem Solving I and Problem Solving II. Available from Inquiry Audio Visuals, 1754 West Farragut Avenue, Chicago, IL 60640. Two sets of four filmstrips. The first includes the titles: Graphs, Solves Problems, Problem Solving: rt=d, the Mathematics of Motion, and Unsolved Problems in Mathematics. The second includes the titles: Word Expressions $\longleftrightarrow$ Algebraic Expressions, Solving Problems: Two Equations, Solving Problems--Guessing, and Problem-Solving: Coin Problems. However, only the filmstrips dealing with graphs, rt=d, word expressions, and guessing appear appropriate for this course. Cost of each set is $27.